10/3/25 Lecture 18

Exam Format

1. Defs
2. Free Response
3. True / False

↳ goal is to find false statements

Example

d) Two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ span $\mathbb{R}^2$ if niether is $\vec{0}$ and $\vec{v}_1 \neq \vec{v}_2$.

False $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

c) $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$T(\vec{x}) = \vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a linear transformation.

False $T(\vec{0}) \neq 0$

if $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then span $(\vec{v}_1, \dots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$.

Q: If $W \subset \mathbb{R}^n$ is a subspace, is $W = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ for some $\vec{v}_1, \dots, \vec{v}_k$?

Proof of Fundamental Theorem of Linear Algebra:

If $\vec{v} \in W$ is non $\vec{0}$, then $\{\vec{v}\}$ is linearly independent.

Moreover any linearly independent subset $\{v_1, \dots, v_k\} \subset W$ has no more than n vectors because $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ and $\mathbb{R}^n = \text{span}(\vec{e}_1, \dots, \vec{e}_n)$ i.e. $k \leq n$ by Lemma

So let $\vec{v}_1, \dots, \vec{v}_k \in W$ be a linearly independent subset with as many vecs as possible.

I claim that $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a basis for W.

To check this I need to show $W = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$

So given $\vec{w} \in W$, I note that $\vec{w}, \vec{v}_1, \dots, \vec{v}_k$ is linearly dependent.

So $\exists$ non-trivial relation $c_1\vec{v}_1 + \dots + c_k\vec{v}_k + c_{k+1}\vec{w} = \vec{0}$

If $c_{k+1} \neq 0$ then $\vec{w} = \frac{-1}{c_{k+1}}(c_1\vec{v}_1 + \dots + c_k\vec{v}_k) \in \text{span}\,\mathcal{B}$.

If $c_{k+1} = 0$ then $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$ must be a non trivial relation among $\vec{v}_1, \dots, \vec{v}_k$.

This contradicts that $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent.

$\therefore$ $\vec{w} \in \text{span}(\vec{v}_1, \dots, \vec{v}_k) = W$ and W has a basis.

Suppose now $\vec{w}_1, \dots, \vec{w}_\ell \in W$ is another basis.

Then our lemma tells us

- $\ell = k$ because $\vec{w}_1, \dots \vec{w}_\ell$ are linearly whereas $\vec{v}_1, \dots, \vec{v}_k$ span W.

and
- $k \leq \ell$ for same reason

$\therefore$ $k = \ell$

$\blacksquare$

Proof of Key lemma:

If $\vec{w} \in W$ is any vector then $\vec{w} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k = V\vec{a}$

$\vec{a} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^k$

$V = [\vec{v}_1 \dots \vec{v}_k]$ = $n \times k$ matrix.

In particular have $\vec{a}_1, \dots, \vec{a}_\ell \in \mathbb{R}^k$ such that $V\vec{a}_1 = \vec{w}_1, \dots, V\vec{a}_\ell = \vec{w}_\ell$. In short:

$$VA = W = [\vec{w}_1 \dots \vec{w}_\ell] = n \times \ell \text{ matrix}$$

$A = [\vec{a}_1 \dots \vec{a}_\ell] = k \times \ell$ matrix

Claim : $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

To check this note $A\vec{x} = \vec{0}$

$\Rightarrow VA\vec{x} = V\vec{0} = \vec{0}$

$\Leftrightarrow W\vec{x} = \vec{0}$

$\Leftrightarrow x_1\vec{w}_1 + \dots + x_k\vec{w}_k = \vec{0}.$

$\Leftrightarrow x_1, \dots x_\ell = 0$ because $\vec{w}_1, \dots, \vec{w}_\ell$ are linearly independent.

Claim implies A is row equivalent to a matrix in RREF with no free variables. which means there is a pivot in each column.

But $\exists$ at most 1 pivot in each row.

$\Rightarrow A$ has at least as many rows as columns. i.e. $k \geq \ell$